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Empirical process methods for classical fiber bundles

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This paper studies the limit distribution for the tensile strength of fiber bundles consisting of parallel and continuous fibers under equal load sharing. The mechanical and statistical behavior of the individual fibers is described by rather general load-strain functions containing random parameters which are motivated by experimental observations of material behavior. The problem is cast within the context of function-indexed empirical processes which provides the structure needed to deduce central limit theorems for classical fiber bundles. This setting allows for a variety of new appliations and generalizations, most of which are unobtainable in the limited context of previous studies. Also, these applications provide a nontrivial example for which empirical process methods are required.

bundles of parallel fibers * random load-strain functions * weak convergence * metric entropy * Vapnik-Chervonenkis class

1. Introduction

This paper places the study of the limit distribution for the strength of classical fiber bundles within the natural context of function-indexed empirical processes and shows that empirical processes theory may be used to deduce central limit theorems (CLT) for fiber bundles consisting of parallel and continuous fibers which satisfy equal load sharing. The empirical processes setting allows for the extension and generalization of known asymptotic results. Equally important, it also provides the structure for new applications of the classical fiber bundle model, first introduced by Daniels (1945). In particular, powerful results from the theory of Vapnik-Chervonenkis (VC) classes of functions as well as metric entropy methods are used. These results from the theory of emprical processes are used to extend and generalize the CLT results of Phoenix and Taylor (1973) and Phoenix (1974, 1975, 1979). Although Phoenix and Taylor (1973) obtained limit theorems for fiber bundles using tools from empirical processes, this paper provides an important generalization of their model to account for more realistic situations and new applications. More exactly, their weak convergence theorems take place in the space $D(0, \infty)$ equipped

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with the Skorohod topology, whereas the limit theorems in this paper take place in the Banach space of bounded functions on $\mathscr{F}$, the underlying parameter set of the empirical process. It is precisely this less restrictive setting which allows for the application of the modern theory of empirical processes.

A classical fiber bundle consists of n parallel continuous fibers. The bundle is clamped at both ends and is stretched by increasing the distance between the clamps. The mechanical failure process is equal load sharing in which all remaining fibers, at any stage, equally share the total applied load. The most general model describes the bundle strength as a function of the mechanical response to the bundle strain. The model includes random variations in the fiber breaking strains, fiber lengths, clamping, or other mechanical properties. The problem is to probabilistically characterize the maximum tensile load that the bundle can sustain as a function of the probabilistic character, the mechanical behavior of the individual fibers, and the failure mechanism.

Let ε be the nominal bundle strain, i.e.,

$$\varepsilon = (L - L_0)/L_0,$$

where L_0 is the reference length of the unstretched and unloaded bundle and where L is the increased bundle length after elongation. Let $Y_i(\varepsilon)$ be the tensile force carried by fiber i when the bundle strain is ε. Assume that $Y_i(\varepsilon)$ can be expressed by

$$Y_i(\varepsilon) = Y_i(\varepsilon, \omega) = q(\varepsilon, \Theta_i(\omega))\mathbf{1}_{(\varepsilon,\infty)}(\xi_i(\omega)), \tag{1.1}$$

where q is a given load-strain function, ξ_i is a nonnegative random scalar denoting the bundle strain at which fiber i breaks, Θ_i is allowed to be a real random vector which may include the variations in fiber slack, clamping, and mechanical properties, and $\mathbf{1}_A(\cdot)$ is the indicator function for the set A. For fixed ω, note that $Y_i(\cdot, \omega)$ represents an element of $D[0, \infty)$, the space of functions which are continuous from the right and have limits from the left (cadlag functions) on $[0, \infty)$.

For fiber i, the random variables ξ_i and Θ_i are allowed to be dependently distributed; however, the random vectors $\{(\Theta_i, \xi_i)\colon\ i \geq 1\}$ are assumed to be independent and identically distributed (i.i.d.). Throughout, $P(\cdot, \cdot)$ denotes the joint c.d.f. for the pair (Θ_i, ξ_i), $\rho(\cdot)$ is the marginal c.d.f. for the breaking strains ξ_i, and $\nu(\cdot)$ is the marginal c.d.f. for Θ_i.

The load-strain function $q(\varepsilon, \theta)$ is assumed to be nonnegative in ε for each θ. This assumption reflects the fact that fibers can support only negligible compressive load. Furthermore, $q(0, \theta)$ is assumed to be identically zero which indicates that zero strain yields zero load supported by the fiber.

Since the bundle consists of parallel fibers, the total tensile force supported by the bundle is the sum of the forces carried by the individual fibers. So that the total load can be considered on a per fiber basis, the normalized bundle load $Q_n(\varepsilon)$ is the total force divided by the number of fibers n in the bundle. Symbolically, $Q_n(\varepsilon)$

is represented by

$$Q_n(\varepsilon) = n^{-1} \sum_{i=1}^{n} Y_i(\varepsilon) \quad \text{for } \varepsilon \text{ in } \mathbb{R}^+.$$

The normalized bundle strength or maximum normalized load Q_n^* is defined as

$$Q_n^* = \sup\{Q_n(\varepsilon)\colon \varepsilon \text{ in } \mathbb{R}^+\},$$

which is the maximum value that the bundle force achieves over the entire range of the bundle strain ε. For physical applications the probabilistic character of Q_n^* is desired.

Many applications are for large bundles in which n may be as large as 10^6, consequently, the main goal is the convergence of Q_n^*. It is prudent first to consider the weak convergence of the sequence of normalized and centered sums $Z_n(\varepsilon)$, defined by

$$Z_n(\varepsilon) = n^{-1/2} \sum_{i=1}^{n} \{Y_i(\varepsilon) - EY_i(\varepsilon)\}, \tag{1.2}$$

to a limiting Gaussian process $Z(\varepsilon)$ which, clearly, has mean zero and covariance

$$\begin{aligned} E[Z(\varepsilon_1)Z(\varepsilon_2)] &= E\{[Y(\varepsilon_1) - EY(\varepsilon_1)][Y(\varepsilon_2) - EY(\varepsilon_2)]\} \\ &= E[Y(\varepsilon_1)Y(\varepsilon_2)] - E[Y(\varepsilon_1)]E[Y(\varepsilon_2)]. \end{aligned} \tag{1.3}$$

The standardized bundle strength is defined by

$$W_n = n^{1/2}[Q_n^* - \mu_{\max}], \tag{1.4}$$

where

$$\mu_{\max} = \sup\{\mu(\varepsilon)\colon \varepsilon \text{ in } \mathbb{R}^+\}$$

and where

$$\mu(\varepsilon) = E[Y_i(\varepsilon)] \quad \text{for } \varepsilon \text{ in } \mathbb{R}^+$$

is the mean load, as a function of the bundle strain ε, carried by fiber i. Since the asymptotic distribution of Q_n^* is readily obtained from the limiting distribution of W_n, it suffices to show the convergence in distribution of W_n to the random variable

$$W = \sup\{Z(\varepsilon)\colon \varepsilon \text{ in } A^*\}, \tag{1.5}$$

where

$$A^* = \{\varepsilon\colon \mu(\varepsilon) = \mu_{\max}\}$$

is the set of values ε at which $\mu(\varepsilon)$ achieves its maximum. Replacing $\mathbb{R}^+$ by its compactified version, if necessary, insures that A^* will be nonempty.

In Section 2 recent general results using metric entropy for empirical processes are used to prove the asymptotic convergence of W_n when q is Lipschitz. The only requirement is that the c.d.f. ρ for the fiber breaking strain ξ_i be continuous with finite $2+\tau$, $\tau>0$, moment. In Section 3 it is shown that if the dependence on the random vector Θ_i in (1.1) is removed, then the elegant theory of empirical processes for VC classes of functions yields an alternative proof for the convergence of W_n when q is bounded and continuous and ρ is continuous. Of greatest significance, however, the new setting involving function-indexed processes permits a variety of applications and generalizations, most of which are unobtainable in the limited context of previous studies; see Section 4.

Notation. Recall that if $\{X_i: i \geq 1\}$ is a sequence of i.i.d. real-valued random variables (r.v.), and $\mathscr{F}$ a class of functions defined on $\mathbb{R}$, then the empirical process $\nu_n(f)$ is defined by

$$\nu_n(f) = n^{-1/2} \sum_{i=1}^{n} \{f(X_i) - Ef(X_i)\} \quad \text{for } f \text{ in } \mathscr{F}. \tag{1.6}$$

Throughout, the envelope $F_{\mathscr{F}}$ for $\mathscr{F}$ is the pointwise supremum of $|f|$ over $\mathscr{F}$. When $F_{\mathscr{F}}$ is finite almost surely with respect to the law of X_i, the stochastic process $\nu_n(f)$ takes its values almost surely in the Banach space $(\ell^\infty(\mathscr{F}), \|\cdot\|_{\mathscr{F}})$ of bounded functions on $\mathscr{F}$, where $\|\cdot\|_{\mathscr{F}}$ denotes the supremum norm

$$\|\cdot\|_{\mathscr{F}} = \sup\{|\cdot(f)|: f \text{ in } \mathscr{F}\}.$$

2. Weak convergence via metric entropy estimations

Let Y_i have the form given in (1.1), and let

$$f\varepsilon(\Theta_i, \xi_i) = q(\varepsilon, \Theta_i(\omega))\mathbf{1}_{(\varepsilon,\infty)}(\xi_i(\omega)),$$

where $f_\varepsilon: \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$f_\varepsilon(s, t) = q(\varepsilon, s)\mathbf{1}_{(\varepsilon,\infty)}(t).$$

If $\mathscr{F} = \{f_\varepsilon: \varepsilon \text{ in } \mathbb{R}^+\}$ and $X_i = (\Theta_i, \xi_i)$, then the weak convergence of $Z_n(\varepsilon)$ defined in (1.2) may be examined by studying the weak convergence of the empirical process $\nu_n(f)$ given in (1.6) within the Banach space $(\ell^\infty(\mathscr{F}), \|\cdot\|_{\mathscr{F}})$. The weak convergence of $Z_n(\varepsilon)$ also could be studied in the space $D[0,\infty)$ equipped with the Skorohod topology (Phoenix and Taylor, 1973), but it is less restrictive to work in the space $(\ell^\infty(\mathscr{F}), \|\cdot\|_{\mathscr{F}})$. The asymptotic behavior of $\nu_n(f)$ in $\ell^\infty(\mathscr{F})$ depends heavily on $\mathscr{F}$ and the c.d.f. P, of course. In general, $\nu_n(f)$, f in $\mathscr{F}$, does not converge weakly in $\ell^\infty(\mathscr{F})$. When $\nu_n(f)$, f in $\mathscr{F}$, does converge weakly, it converges to a mean zero

Gaussian process $G_P(f)$ with covariance

$$EG_P(f_{\varepsilon_1})G_P(f_{\varepsilon_2}) = \int f_{\varepsilon_1} f_{\varepsilon_2}\,\mathrm{d}P - \int f_{\varepsilon_1}\,\mathrm{d}P \int f_{\varepsilon_2}\,\mathrm{d}P$$

$$= E[Y(\varepsilon_1)Y(\varepsilon_2)] - E[Y(\varepsilon_1)]E[Y(\varepsilon_2)], \tag{2.1}$$

which is identical to the covariance function (1.3).

Theorem 2.1. *The process* $\{Z_n(\varepsilon): \varepsilon \text{ in } \mathbb{R}^+\}$ *defined in* (1.2) *converges weakly to a mean zero Gaussian process* $Z(\varepsilon) = G_P(f_\varepsilon)$ *in* $\ell^\infty(\mathscr{F})$ *with covariance function given by* (2.1) *for any q which is Lipschitz continuous in the first argument and for any joint c.d.f. P for* (Θ_i, ξ_i) *which has a marginal c.d.f.* ρ *for* ξ_i *with finite* $2+\tau$, $\tau > 0$, *moment. If, in addition,* ρ *is continuous, then* W_n *defined in* (1.4) *converges in distribution to W of* (1.5).

It should be noted that since q is Lipschitz continuous it satisfies the following condition for the first argument:

$$|q(\varepsilon_1, \Theta_i) - q(\varepsilon_2, \Theta_i)| \leq C|\varepsilon_1 - \varepsilon_2| \quad \text{for all } \varepsilon_1 \text{ and } \varepsilon_2, \tag{2.2}$$

where C is a constant independent of Θ_i. Notice that (2.2) and the assumption that $q(0, \Theta_i) = 0$ imply $q(\varepsilon, \Theta_i) \leq C\varepsilon$ for ε in $\mathbb{R}^+$.

To prove Theorem 2.1, the recently developed limit theory for general function-indexed empirical processes (1.6) will be used. First, recall the following definition and theorem:

Definition 2.2. Let $\mathscr{F}$ be a class of functions on $\mathbb{R}^2$ and P a probability measure on $\mathbb{R}^2$. The $\mathscr{L}^2(P)$ *bracketing number* (or *covering number*) for $\mathscr{F}$ is defined for all $\delta > 0$ as $N_{[\,]}(\delta, \mathscr{F}, P) = \inf\{n$: there exist measurable $f_1, \ldots, f_n$ such that for all f in $\mathscr{F}$ there are $i, j \leq n$ with $f_i \leq f \leq f_j$ pointwise and $(\int |f_i - f_j|^2\,\mathrm{d}P)^{1/2} \leq \delta\}$.

As a remark, log $N_{[\,]}(\delta, \mathscr{F}, P)$ is called *metric entropy with bracketing* and was first introduced by Dudley (1978) as a tool for studying the weak convergence of set-indexed empirical processes. The following theorem from Ossiander (1987) illustrates the applicability of metric entropy in the context of central limit theorems for empirical processes.

Theorem 2.3. *Let* $(X, \mathscr{A}, P)$ *be a probability space; let* $\mathscr{F}$ *be contained in* $\mathscr{L}^2(X, \mathscr{A}, P)$ *with an envelope function in* $\mathscr{L}^2(X, \mathscr{A}, P)$. *Suppose that*

$$\int_0^1 \{\log N_{[\,]}(\delta, \mathscr{F}, P)\}^{1/2}\,\mathrm{d}\delta < \infty.$$

Then $\nu_n(f)$ converges weakly to a mean zero Gaussian process $G_P(f)$ with covariance given by (2.1). $\square$

Equipped with this theorem sufficient conditions guaranteeing the weak convergence of $\nu_n(f)$, f in $\mathscr{F}$, can be obtained. The following lemma establishes crucial polynomial bounds on the covering number $N_{[\,]}$ for the class $\mathscr{F}$.

Lemma 2.4. *Let $\mathscr{F}=\{f_\varepsilon:\ \varepsilon$ in $\mathbb{R}^+\}$ and P be the joint c.d.f. for (Θ_i, ξ_i). If ρ, the marginal c.d.f. for ξ_i, has finite $2+\tau$ moment for some $\tau>0$, then there exists a $K<\infty$ such that $N_{[\,]}(\delta, \mathscr{F}, P)=\mathrm{O}(\delta^{-K})$.*

Proof. The proof has three steps, each of which is straightforward.

(i) Recall that $q(\varepsilon, s)\leqslant C\varepsilon$ for all $\varepsilon\geqslant 0$ because q is Lipschitz and $q(0, s)=0$. Since ρ has finite $2+\tau$ moment, it follows that given $\delta>0$, there is an $M=M(\delta)<\infty$ such that

$$\int_{t\geqslant M} t^2\,\mathrm{d}\rho(t)<\delta^2/C^2.$$

By the moment hypothesis, it may be shown by means of Hölder's inequality applied to the r.v. $\xi_i^2\mathbf{1}_{(\xi_i\geqslant M)}$ that there exists a finite constant K depending only on τ such that $M(\delta)=\mathrm{O}(\delta^{-K})$.

(ii) Let f_ε be as above. In this step, the metric entropy for the class of functions f_ε, where $\varepsilon\geqslant M$, is evaluated. By considering the functions $f^{\mathrm{L}}(s, t)=0$ and $f^{\mathrm{U}}(s, t)=Ct\mathbf{1}_{(M,\infty)}(t)$, it is clear that, *for all $\varepsilon\geqslant M$, $f^{\mathrm{L}}\leqslant f_\varepsilon\leqslant f^{\mathrm{U}}$*. The first inequality is obvious, and the second follows from

$$f_\varepsilon(s, t)\leqslant C\varepsilon\mathbf{1}_{(\varepsilon,\infty)}(t)\leqslant Ct\mathbf{1}_{(\varepsilon,\infty)}(t)\leqslant Ct\mathbf{1}_{(M,\infty)}(t).$$

Moreover, by the choice of M in step (i),

$$\int\!\!\int (f^{\mathrm{U}}-f^{\mathrm{L}})^2\,\mathrm{d}P\leqslant C^2\int_{t\geqslant M} t^2\,\mathrm{d}\rho(t)<\delta^2.$$

Therefore, the covering number $N_{[\,]}$ for those functions f_ε where $\varepsilon\geqslant M$ is simply equal to 2.

(iii) It suffices to show that the covering number $N_{[\,]}$ for the remaining subclass of functions $\{f_\varepsilon: 0\leqslant\varepsilon<M\}$ is $\mathrm{O}(M/\delta^{-D})$ for some $D<\infty$. To show this, let ε_i, $1\leqslant i\leqslant 2\sqrt{2}\,CM/\delta$, be a $\delta/2\sqrt{2}C$-partition of the interval $[0, M]$. Consider bracketing functions of the form

$$f_i^{\mathrm{U}}(s, t)=\{q(\varepsilon_i, s)+\delta/2\sqrt{2}\}\mathbf{1}_{(\varepsilon_i,\infty)}(t)$$

and

$$f_i^{\mathrm{L}}(s, t)=\{q(\varepsilon_i, s)-\delta/2\sqrt{2}\}\mathbf{1}_{(\varepsilon_{i+1},\infty)}(t).$$

If $\varepsilon_i < \varepsilon < \varepsilon_{i+1}$, then it is easily seen that the Lipschitz condition on q implies

$$f_i^{\mathrm{L}} \leqslant f_\varepsilon \leqslant f_i^{\mathrm{U}}.$$

Also, note that

$$\iint (f_i^{\mathrm{U}} - f_i^{\mathrm{L}})^2 \,\mathrm{d}P(s,t)$$

$$= \int_{\varepsilon_i}^{\varepsilon_{i+1}} \int_{-\infty}^{+\infty} (f_i^{\mathrm{U}} - f_i^{\mathrm{L}})^2 \,\mathrm{d}P(s,t) + \int_{\varepsilon_{i+1}}^{+\infty} \int_{-\infty}^{+\infty} (f_i^{\mathrm{U}} - f_i^{\mathrm{L}})^2 \,\mathrm{d}P(s,t)$$

$$= \int_{\varepsilon_i}^{\varepsilon_{i+1}} \int_{-\infty}^{+\infty} [q(\varepsilon_i, s) + \delta/2\sqrt{2}]^2 \,\mathrm{d}P(s,t) + \int_{\varepsilon_{i+1}}^{+\infty} \int_{-\infty}^{+\infty} \tfrac{1}{2}\delta^2 \,\mathrm{d}P(s,t)$$

$$\leqslant \int_{\varepsilon_i}^{\varepsilon_{i+1}} \int_{-\infty}^{\infty} (CM + \delta/2\sqrt{2})^2 \,\mathrm{d}P(s,t) + \int_{\varepsilon_{i+1}}^{+\infty} \int_{-\infty}^{+\infty} \tfrac{1}{2}\delta^2 \,\mathrm{d}P(s,t)$$

$$= \int_{\varepsilon_i}^{\varepsilon_{i+1}} \tfrac{1}{2}(\sqrt{2}\,CM + \tfrac{1}{2}\delta)^2 \,\mathrm{d}\rho(t) + \int_{\varepsilon_{i+1}}^{+\infty} \tfrac{1}{2}\delta^2 \,\mathrm{d}\rho(t),$$

since for $\varepsilon \leqslant M$, $q(\varepsilon_i, s) \leqslant CM$. When ρ is continuous, the partition $\{\varepsilon_i\}$ may be further refined to obtain a new partition $\{\varepsilon_i'\}$ with the property that $\rho([\varepsilon_i', \varepsilon_{i+1}']) \leqslant \delta^2/(\sqrt{2}\,CM + \frac{1}{2}\delta)^2$ for all i. Note that the cardinality of $\{\varepsilon_i'\}$ is at most the cardinality of $\{\varepsilon_i\}$ multiplied by a factor of $(\sqrt{2}\,CM + \frac{1}{2}\delta)^2/\delta^2$. Thus, the right-hand side is bounded by δ^2, as desired. When ρ is discontinuous with jumps of magnitude greater than $\delta^2/(\sqrt{2}\,CM + \frac{1}{2}\delta)^2$ at points $\{e_j\}_{j\geqslant 1}$, the above refining procedure breaks down. When this occurs, only the additional set of bracketing functions of the form

$$f_j^{\mathrm{L}}(s,t) = f_j^{\mathrm{U}}(s,t) = q(e_j, s)\mathbf{1}_{(e_j,\infty)}(t)$$

needs to be considered. The cardinality of $\{e_j\}$ is finite and at most $(\sqrt{2}\,CM + \frac{1}{2}\delta)^2/\delta^2$. Then the above refining procedure may be repeated with respect to the measure ρ' defined by

$$\rho' = \rho - (\textstyle\sum \delta_{e_j}).$$

In either event the number of bracketing functions is $\mathrm{O}(M/\delta^{-D})$ where D is a finite constant. $\square$

Combining Lemma 2.4 with Theorem 2.3, it is now manifest that

$$n^{-1/2} \sum_{i=1}^{n} \{Y_i(\varepsilon) - EY_i(\varepsilon)\} \xrightarrow{\mathrm{w}} G_P(f_\varepsilon),$$

whenever q is Lipschitz continuous in the first argument and ρ has a finite $2+\tau$,

$\tau > 0$, moment. Note that $\mathscr{F}$ has an envelope in $\mathscr{L}^2(P)$ since

$$\iint \left[\sup_{\varepsilon \geq 0} q(\varepsilon, s)\mathbf{1}_{(\varepsilon,\infty)}(t)\right]^2 \mathrm{d}P(s, t)$$

$$\leq \iint \sup_{\varepsilon \geq 0} C^2\varepsilon^2\left[\mathbf{1}_{(\varepsilon,\infty)}(t)\right]^2 \mathrm{d}P(s, t)$$

$$\leq C^2 \iint t^2\,\mathrm{d}P(s, t) = C^2 \int t^2\,\mathrm{d}\rho(t) < \infty.$$

This concludes the proof of the first part of Theorem 2.1.

The proof of the second part of Theorem 2.1 will be achieved through the continuous mapping theorem. Let $\mathscr{C}(\mathscr{F}, P)$ signify the collection of uniformly continuous functions on $(\mathscr{F}, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the $\mathscr{L}^2(P)$ seminorm on $\mathscr{F}$. If $Z_n(\varepsilon)$ converges weakly to a limiting Gaussian process $Z(\varepsilon) = G_P(f_\varepsilon)$, then G_P almost surely has values in $\mathscr{C}(\mathscr{F}, P)$ (Pollard, 1984, p. 156). In other words, for every $\beta > 0$, there exists a $\delta > 0$ such that

$$\text{if } \|f_{\varepsilon_1} - f_{\varepsilon_2}\|_2 < \delta, \quad \text{then } |G_P(f_{\varepsilon_1}) - G_P(f_{\varepsilon_2})| < \beta \quad \text{almost surely.} \tag{2.3}$$

It should be noted that Phoenix and Taylor (1973) assumed that ρ was absolutely continuous with a finite sixth moment. By contrast, the assumption needed for this development is that ρ is simply continuous with a finite $2+\tau$, $\tau > 0$, moment. Furthermore, the next lemma shows that under various sets of additional assumptions on q, $G_P(f_\varepsilon)$ actually has values almost surely in the set Λ of bounded functions on $\mathscr{F} = \{f_\varepsilon\colon \varepsilon \text{ in } \mathbb{R}^+\}$ which are continuous with respect to the indexing set $\mathbb{R}^+$ endowed with the usual topology. This is of critical importance in proving the upcoming Lemma 2.6 which is based on the continuous mapping theorem.

At this stage of the development cases (i) and (iii) considered in Lemma 2.5 below are not needed. However, they are needed for the results of Sections 3 and 4, where it is necessary to know that the limiting Gaussian processes associated with the function q of cases (i) and (iii) almost surely have values in Λ.

Lemma 2.5. *Let $\mathscr{F}$ be as above and assume that ρ, the c.d.f. for ξ_i, is continuous. Suppose that q and ρ satisfy one of the following*:

(i) *$q(\varepsilon, s)$ is bounded and continuous in ε, uniformly in s, i.e., for any $\alpha > 0$ there is a $\delta > 0$ such that for any s, $|q(\varepsilon_1, s) - q(\varepsilon_2, s)| < \delta$ whenever $|\varepsilon_1 - \varepsilon_2| < \alpha$, or*

(ii) *$q(\varepsilon, s)$ is Lipschitz in ε, $q(0, s) = 0$; ρ has a finite second moment, or*

(iii) *$q(\varepsilon, s)$ is increasing in ε, continuous in ε, uniformly in s, and $\sup_{s\geq 0} q(t, s)$ is in $\mathscr{L}^2(\rho)$.*

Then $G_P(f_\varepsilon)$ almost surely has values in Λ.

Proof. It must be shown that for every $\varepsilon > 0$ and for every $\beta > 0$ there exists $\alpha > 0$ such that

$$|\varepsilon - \varepsilon_1| < \alpha \quad \text{implies} \quad |G_P(f_\varepsilon) - G_P(f_{\varepsilon_1})| < \beta \quad \text{almost surely.} \tag{2.4}$$

To prove (2.4), it suffices by (2.3) to show that given $\delta>0$, there is an α small enough such that $|\varepsilon-\varepsilon_1|<\alpha$ implies $\|f_{\varepsilon_1}-f_{\varepsilon_2}\|_2<\delta$. This implication follows easily from the triangle inequality since

$$\begin{aligned}
&\|q(\varepsilon, s)\mathbf{1}_{(\varepsilon,\infty)}(t)-q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon_1,\infty)}(t)\|_2\\
&\quad\leqslant \|q(\varepsilon, s)\mathbf{1}_{(\varepsilon,\infty)}(t)-q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon,\infty)}(t)\|_2\\
&\qquad+\|q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon,\infty)}(t)-q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon_1,\infty)}(t)\|_2.
\end{aligned}$$

Note that the first term can be made small by the assumed continuity of q. The second term can be made small by either (i) boundedness of q and continuity of ρ for condition (ii) since, assuming without loss of generality that $\varepsilon_1<\varepsilon$:

$$\begin{aligned}
&\|q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon,\infty)}(t)-q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon_1,\infty)}(t)\|_2\\
&\quad\leqslant \int_{\varepsilon_1}^{\varepsilon}\int_{-\infty}^{+\infty} C^2\varepsilon_1^2\,\mathrm{d}P(s,t)=\int_{\varepsilon_1}^{\varepsilon} C^2\varepsilon_1^2\,\mathrm{d}\rho(t)\leqslant\int_{\varepsilon_1}^{\varepsilon} C^2t^2\,\mathrm{d}\rho(t)\leqslant\tfrac{1}{2}\delta,
\end{aligned}$$

by the assumed continuity and second moment hypothesis on ρ. Condition (iii) also implies that the second term can be made small, since if $|\varepsilon-\varepsilon_1|$ is small enough then

$$\begin{aligned}
&\iint |q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon,\infty)}(t)-q(\varepsilon_1, s)\mathbf{1}_{(\varepsilon_1,\infty)}(t)|^2\,\mathrm{d}P(s,t)\\
&\quad\leqslant\int_{\varepsilon_1}^{\varepsilon}\int_{-\infty}^{+\infty}[q(\varepsilon_1,s)]^2\,\mathrm{d}P(s,t)\leqslant\int_{\varepsilon_1}^{\varepsilon}\int_{-\infty}^{+\infty}[q(t,s)]^2\,\mathrm{d}P(s,t)\\
&\quad\leqslant\int_{\varepsilon_1}^{\varepsilon}\sup_{s\geqslant 0}[q(t,s)]^2\,\mathrm{d}\rho(t)\leqslant\tfrac{1}{2}\delta.
\end{aligned}$$

The second inequality uses the monotonicity of $q(\cdot, s)$, and the final inequality follows since $\sup_{s\geqslant 0} q(t,s)$ is necessarily increasing, in $\mathscr{L}^2(\rho)$, and ρ is continuous. □

The next lemma completes the proof of Theorem 2.1. This lemma implicitly assumes the weak convergence of $Z_n(\varepsilon)$ to $G_P(f_\varepsilon)$. Later it will be shown that, under any of the three conditions of Lemma 2.5, weak convergence actually holds.

Lemma 2.6. *Let q and ρ satisfy any of the three conditions of Lemma 2.5. Then $W_n=n^{1/2}[Q_n^*-\mu_{\max}]$ converges in distribution to $W=\sup\{Z(\varepsilon)\colon \varepsilon \text{ in } A^*\}$.*

Proof. The proof is obtained by making some modifications to Proposition 4 of Phoenix and Taylor (1973) and by using Lemma 2.5. Instead of considering $D[0,\infty)$ equipped with the Skorohod topology, the Banach space $(\ell^\infty(\mathscr{F}), \|\cdot\|_{\mathscr{F}})$ is used. By Lemma 2.5, the limiting Gaussian process G_P is contained in the subset A of $\ell^\infty(\mathscr{F})$ with probability one.

To show that W_n converges in distribution to W, the continuous mapping theorem is invoked (Billingsley, 1968, Theorem 5.5). Introduce the functions h_n and h on $\ell^\infty(\mathscr{F})$ defined for ω in $\ell^\infty(\mathscr{F})$ by

$$h_n(\omega) = \sup_{\varepsilon \geq 0}\{\omega(f_\varepsilon) - n^{1/2}[\mu_{\max} - \mu(\varepsilon)]\} \tag{2.5}$$

and

$$h(\omega) = \sup\{\omega(f_\varepsilon): \varepsilon \text{ in } A^*\}.$$

Observe that $h_n(Z_n) = W_n$ and $h(Z) = W$, and it thus suffices to show that $h_n(Z_n) \xrightarrow{w} h(Z)$. By the continuous mapping theorem and Lemma 2.5, it suffices to show $\lim_{n\to\infty} h_n(\omega_n) = h(\omega)$ whenever ω is in Λ and ω_n is a sequence of functions in $\ell^\infty(\mathscr{F})$ converging to ω in the norm $\|\cdot\|_{\mathscr{F}}$. However, this follows from Proposition 4 of Phoenix and Taylor (1973) by replacing $\omega(t)$, $0 \leq t \leq 1$, by $\omega(f_\varepsilon)$, ε in $\mathbb{R}^+$, and from the fact that the supremum in (2.5) is actually attained, since ω in Λ is in effect a continuous function on the compactified half line endowed with the usual topology. This completes the proof of Lemma 2.6. □

3. Weak convergence via VC classes of functions

Now, consider a simplified form of (1.1) for Y_i. Let

$$f_\varepsilon(\xi_i) = q(\varepsilon)\mathbf{1}_{(\varepsilon,\infty)}(\xi_i), \tag{3.1}$$

where $f_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$f_\varepsilon(t) = q(\varepsilon)\mathbf{1}_{(\varepsilon,\infty)}(t).$$

Notice that the dependence on the random vector Θ_i has been removed. This simplification corresponds to a deterministic load-strain function for each fiber, but with a breaking strain which is random. For (3.1) in (1.2) the following CLT can be established.

Theorem 3.1. *The process $\{Z_n(\varepsilon)$: ε in $\mathbb{R}^+\}$ converges weakly to a mean zero Gaussian process $Z(\varepsilon)$ in $\ell^\infty(\mathscr{F})$ with covariance function given by* (2.1) *for any bounded q and for arbitrary ρ. If, in addition, q and ρ are continuous, then W_n converges in distribution to W.*

The proof of Theorem 3.1 depends upon the fact that for classes of functions having the *VC property*, $\nu_n(f)$ in the space $\ell^\infty(\mathscr{F})$ will *always* converge weakly to a Gaussian process for *any* choice of ρ. To understand the VC property, consider, first, some basic definitions concerning VC classes of sets:

Definition 3.2. A class $\mathscr{D}$ of subsets of a set S *shatters* a set B if and only if all subsets of B are intersections of sets in $\mathscr{D}$ with B.

Definition 3.3. The class $\mathscr{D}$ is called VC if and only if there exists an $n < \infty$ such that no n element subset of S is shattered by $\mathscr{D}$.

Next, given $\mathscr{F}$, let $\mathscr{G}$ be the collection of graphs formed by $\mathscr{F}$, that is, $\mathscr{G}$ is the collection of all sets $\mathcal{g}$ of the form

$$\mathcal{g} = \{(x, y): y \leq f(x)\} \quad \text{for } f \text{ in } \mathcal{F}.$$

The class $\mathcal{F}$ has the *VC property* if and only if $\mathcal{G}$ is a *VC class of sets.*

The next result, taken from Dudley (1984), is essential to the following. This theorem neither includes nor is included in Theorem 2.3.

Theorem 3.4. *Let $\mathcal{F}$ be a class of bounded functions on $\mathbb{R}^+$ with the VC property. Let*

$$\nu_n(f) = n^{-1/2} \sum_{i=1}^{n} \{f(X_i) - Ef(X_i)\},$$

where the law of X_i is ρ. Then $\nu_n(f)$ converges weakly to $G_\rho(f)$, where $G_\rho(f)$ is a mean zero Gaussian process with covariance given by (2.1), *where P is replaced by ρ.* □

Remark. If $\mathcal{F}$ is not bounded, then the above CLT still holds whenever the envelope $F_{\mathcal{F}}$ for $\mathcal{F}$ is in $\mathcal{L}^2(\rho)$ (Pollard, 1984).

Notice that no assumptions are made on the underlying c.d.f. ρ in Theorem 3.4.

Proof of Theorem 3.1. The normalized sums $Z_n(\varepsilon)$ in (1.2) have the form

$$Z_n(\varepsilon) = n^{-1/2} \sum_{i=1}^{n} \{f_\varepsilon(\xi_i) - Ef_\varepsilon(\xi_i)\} \quad \text{for } \varepsilon \text{ in } \mathbb{R}^+,$$

or, equivalently,

$$Z_n(\varepsilon) = n^{-1/2} \sum_{i=1}^{n} \{f_\varepsilon(\xi_i) - Ef_\varepsilon(\xi_i)\} = \nu_n(f_\varepsilon) \quad \text{for } f_\varepsilon \text{ in } \mathcal{F},$$

where $\mathcal{F}$ denotes the class of functions

$$\mathcal{F} = \{t \to q(\varepsilon)\mathbf{1}_{(\varepsilon,\infty)}(t): \varepsilon \text{ in } \mathbb{R}^+\}.$$

It suffices to show that $\mathcal{F}$ has the VC property, for then $\nu_n(f_\varepsilon)$ converges weakly to $G_\rho(f_\varepsilon)$ which has covariance given in (2.1) with ρ in place of P. It is easily checked that the graphs of functions in $\mathcal{F}$ do form a VC class. Indeed, inspection of f_ε shows that the graphs are *rectangular slabs* with infinite extent to the right, and it is manifest that such a class of sets will not shatter a three point set. The convergence of W_n to W follows from Lemmas 2.5 and 2.6. Thus, the proof of Theorem 3.1 is completed. □

Remarks. (1) If q is unbounded, but increasing and in $\mathcal{L}^2(\rho)$, where ρ is continuous, then Theorem 3.1 still holds via Lemmas 2.5, (iii) and 2.6 and the remark following Theorem 3.4. For this case, the envelope $F_{\mathcal{F}}$ is simply q, which is in $\mathcal{L}^2(\rho)$.

(2) Theorem 2.1 cannot be proved using VC techniques, since the relevant class of functions there is not necessarily a VC class.

4. Generalizations of the weak convergence and additional applications

In this section, significant and realistic applications of both the metric entropy techniques and the VC theory of Sections 2 and 3, respectively, are developed for some of the important physical processes. Phoenix and Taylor (1973), Phoenix (1974, 1975, 1979), Smith (1982), and Daniels (1989) applied their results to applications arising from yarns and cables. Some of the applications considered generalize their examples by relaxing their restrictions on P. More important, the setting of function-indexed empirical processes yields a variety of new applications and generalizations, many of which are unobtainable within the limited scope of earlier work. In effect, all of the applications below are extensions or corollaries of Theorems 2.1, 3.1 and 3.4.

Random slack

Let Θ_i be the random slack in fiber i, i.e.,

$$\Theta_i = (U_i - L_0)/L_0,$$

where U_i is the unstretched and unloaded length of fiber i. Assume that $\{\Theta_i : i \geq 1\}$ are i.i.d. nonnegative r.v.'s. The actual strain ε_i' in fiber i is nonzero only after the nominal strain ε has exceeded the slack Θ_i, i.e.,

$$\varepsilon_i' = \max\{(\varepsilon - \Theta_i), 0\}.$$

Thus, the nonnegative, load-strain function q in (1.1) is simply a translation expressed as

$$q(\varepsilon, \theta) = q[\max\{(\varepsilon - \theta), 0\}]. \tag{4.1}$$

Let ξ_i' be the actual breaking strain of fiber i, then the nominal breaking strain ξ_i is given by

$$\xi_i = \xi_i' + \Theta_i,$$

and, naturally, the random tensile force is given by

$$Y_i(\varepsilon) = q[\max\{(\varepsilon - \Theta_i), 0\}]\mathbf{1}_{(\varepsilon,\infty)}(\xi_i). \tag{4.2}$$

Since Θ_i is simply the effect of unequal lengths of fibers and not the effect of a material property, then, physically, it is most reasonable to assume that Θ_i and ξ_i' are independent; however, this assumption is not necessary for the following development.

Since q in (4.1) is a translate, then the relevant indexing class $\mathscr{F}$ of functions is given by the truncated translates of q, i.e., for (s, t) in $\mathbb{R}^+ \times \mathbb{R}^+$,

$$\mathscr{F} = \{(s, t) \to q(\varepsilon - s)\mathbf{1}_{(\varepsilon,\infty)}(t) \colon \varepsilon \text{ in } \mathbb{R}^+\}.$$

It is known that whenever q is of bounded variation, the translates of q have the VC property (Pollard, 1984, p. 42). Furthermore, truncations from below of the graphs formed by the class of translates by the indicator function $\mathbf{1}_{(\varepsilon,\infty)}(\cdot)$ preserve the VC property. This follows since graphs of truncations are merely the intersections of the graphs of translates (a VC class) with graphs of indicators over half planes (also a VC class) and the fact that VC classes are stable under a finite number of Boolean operations (Dudley, 1978, Proposition 7.12). Thus, by Theorem 3.4, the weak convergence of Z_n with the force functions of (4.2) is assured without any further assumptions on the marginal c.d.f.'s for ξ_i' and Θ_i. If q and ρ are continuous, then W_n converges in distribution to W by Lemmas 2.5(i) and 2.6.

If q is not of bounded variation but Lipschitz as in (2.2), then Theorem 2.1 can be used under the additional assumptions that the marginal c.d.f. ρ for ξ_i' has a finite $2+\tau$, $\tau>0$, moment and that ρ be continuous.

A generalization of random slack in (4.2) is to modify the force functions as follows:

$$Y_i(\varepsilon)=q(\varepsilon)\mathbf{1}_{[0,\varepsilon]}(\zeta_i)\mathbf{1}_{(\varepsilon,\infty)}(\xi_i), \tag{4.3}$$

where $\{X_i=(\zeta_i, \xi_i): i\geqslant 1\}$ is a sequence of i.i.d. random vectors for which the components may be dependent. Again, ξ_i is the breaking strain, but ζ_i is a random strain for which the load carried by fiber i is zero for smaller strains and for which there is a jump of magnitude $q(\zeta_i)$ in the load at ζ_i. As it stands, if q is bounded and continuous, then Theorem 3.1 may be readily adapted for the equivalent representation of the force functions given by

$$Y_i(\varepsilon)=q(\varepsilon)\mathbf{1}_{[0,\varepsilon]\times(\varepsilon,\infty)}(X_i),$$

provided that the c.d.f. of X_i is continuous. Note that the functions given by

$$f_\varepsilon(X_i)=q(\varepsilon)\mathbf{1}_{[0,\,\varepsilon]\times(\varepsilon,\infty)}(X_i)$$

form a VC class of functions since rectangular solid slabs form a VC class of sets.

If q in (4.3) is further generalized to include a dependency upon the r.v. Θ_i as in (1.1), then Theorem 2.1 can be applied whenever the joint c.d.f. for $(\Theta_i, \zeta_i, \xi_i)$ has a continuous marginal c.d.f. for ξ_i with a finite $2+\tau$ moment and q is Lipschitz in the first argument.

Polynomial behavior

Theorem 3.1 may be extended to force-strain functions of the form given by (1.1) where q is a fixed polynomial in its two arguments and where $\{(\Theta_i, \xi_i): i\geqslant 1\}$ is a sequence of i.i.d. random vectors with possibly dependent components. The collection of functions $\mathscr{F}$ which is relevant for this application is given by

$$\mathscr{F}=\{(s,t)\to q(\varepsilon,s)\mathbf{1}_{(\varepsilon,\infty)}(t):\ \varepsilon \text{ in } \mathbb{R}^+\}.$$

To see that $\mathscr{F}$ is a VC class of functions it suffices to show that the class

$$\{(s,t)\to q(\varepsilon,s):\ \varepsilon \text{ in } \mathbb{R}^+\}$$

is VC, i.e., to show that the collection $\mathscr{G}$ of graphs

$$\mathscr{G}=\{\{(s,t,w): q(\varepsilon,s,t)\leqslant w\}:\ \varepsilon \text{ in } \mathbb{R}^+\}$$

is a VC class of sets. Notice that

$$\mathscr{G}=\{\{(s, t, w): P_\varepsilon(s, t, w)>0\}: \varepsilon \text{ in } \mathbb{R}^+\},$$

where $P_\varepsilon(s, t, w)=w-q(\varepsilon, s, t)$ is a fixed polynomial of bounded degree, i.e., there is a constant $K<\infty$ such that, for all ε in $\mathbb{R}^+$, $\deg(P_\varepsilon(s, t, w))\leq K$. Therefore, $\mathscr{G}$ is a subset of the set $\mathscr{C}$ of all positivity sets of polynomials of bounded degree in (s, t, w). Since the class $\mathscr{C}$ is known to be VC (Dudley, 1978) and since subclasses of VC classes are VC, it follows that $\mathscr{G}$ is VC. Thus, $\mathscr{F}$ is a VC class of functions.

Now consider the weak convergence of the $Z_n(\varepsilon)$ of (1.2). If $\{\xi_i: i\geq 1\}$ is a bounded collection of r.v.'s, say $|\xi_i|\leq B$, then $Y_i(\varepsilon)$ is nonzero only if $0\leq\varepsilon\leq B$. Hence, the envelope $F_{\mathscr{F}}$ for $\mathscr{F}$ is

$$F_{\mathscr{F}}(s, t)=\sup_{\varepsilon}|q(\varepsilon, s)\mathbf{1}_{(\varepsilon,\infty)}(t)|=\sup_{0\leq\varepsilon\leq B}|q(\varepsilon, s)|.$$

If Θ_i is also bounded, then $F_{\mathscr{F}}$ is uniformly bounded and clearly in $\mathscr{L}^2(P)$, and the desired weak convergence of $Z_n(\varepsilon)$ holds by Theorem 3.4, where P is to be substituted for ρ. Since $q(\varepsilon, s)$ is continuous in ε, uniformly in s, and bounded, W_n converges to W by Lemmas 2.5(i) and 2.6, if, in addition to being bounded, ξ_i is continuous. As a final comment, it is manifest that q could be a fixed polynomial in any finite number of bounded r.v.'s and the result would follow.

As a passing comment, note that it would be difficult to demonstrate the metric entropy condition for general polynomial classes of functions. Thus, here is an example where VC theory is applicable, but metric entropy is not.

Based upon empirical observations, it has been known for some time that many metals exhibit a stress-strain behavior under tensile loading that can be represented by a simple polynomial, at least over the plastic region (Smith, 1950). The following polynomial form for the load-strain function is the most frequently used:

$$Y_i(\varepsilon)=q(\varepsilon, \phi, \lambda)\mathbf{1}_{(\varepsilon,\infty)}(\xi_i)=\phi\varepsilon^\lambda\mathbf{1}_{(\varepsilon,\infty)}(\xi_i),$$

where ϕ and λ are material parameters which are usually taken to be constants. Obviously, the weak convergence for this application follows directly from the above comments.

A general load-strain response

A more realistic form of $Y_i(\varepsilon)$, which allows Φ_i to be a r.v., would be

$$Y_i(\varepsilon)=q_1(\varepsilon, \Theta_i)\mathbf{1}_{(\varepsilon,\infty)}(\eta_i)+q_2(\varepsilon, \eta_i, \Phi_i)\mathbf{1}_{[0,\varepsilon]}(\eta_i)\mathbf{1}_{(\varepsilon,\infty)}(\xi_i), \tag{4.4}$$

where η_i is the random strain at which transition between the two load-strain behaviors occurs, ξ_i is the random breaking strain, and Θ_i and Φ_i are random material parameters appropriate to the first and second portions, respectively, of the load-strain function. Note that q_2 must be a function of η_i in order to guarantee the continuity of Y_i at η_i.

Assume that q_1 in (4.4) satisfies appropriate conditions so that

$$\mathscr{F}_1 = \{(r, s, t, u) \to q_1(\varepsilon, s)\mathbf{1}_{(\varepsilon,\infty)}(r) \colon \varepsilon \text{ in } \mathbb{R}^+\}$$

is a VC collection of functions, e.g., q_1 could satisfy the conditions of Theorem 3.1, which is independent of Θ_i, the random slack application, or the polynomial application. Let q_2 fulfill the conditions for the polynomial behavior described in the above paragraph so that

$$\mathscr{F}_2 = \{(r, s, t, u) \to q_2(\varepsilon, r, u)\mathbf{1}_{[0,\varepsilon]\times(\varepsilon,\infty)}(r, t) \colon \varepsilon \text{ in } \mathbb{R}^+\}$$

is also a VC collection of functions. Then the collection of functions $\mathscr{F}$ which is appropriate for this application is given by

$$\mathscr{F} = \{(r, s, t, u) \to q_1(\varepsilon, s)\mathbf{1}_{(\varepsilon,\infty)}(r) + q_2(\varepsilon, r, u)\mathbf{1}_{[0,\varepsilon]\times(\varepsilon,\infty)}(r, t) \colon \varepsilon \text{ in } \mathbb{R}^+\},$$

where (r, s, t, u) is in $(\mathbb{R}^+)^4$, and notice that $\mathscr{F} = \{f_1 + f_2 \colon f_1 \text{ in } \mathscr{F}_1, f_2 \text{ in } \mathscr{F}_2\}$. Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are both VC classes of functions the empirical process $\nu_n(f)$ in (1.6), for f in $\mathscr{F}$, satisfies the CLT whenever the envelope $F_{\mathscr{F}}$ is in $\mathscr{L}^2(P)$, i.e.,

$$n^{-1/2} \sum_{i=1}^{n} \{Y_i(\varepsilon) - EY_i(\varepsilon)\}, \quad \varepsilon \text{ in } \mathbb{R}^+,$$

converges weakly, as desired (Pollard, 1982, Theorem 10, part (i)). The application is complete via Lemmas 2.5 and 2.6 as long as q, ξ_i, and η_i are continuous.

Random modulus

For fibers which exhibit linearly elastic load-strain behavior the appropriate function is

$$q(\varepsilon, \theta) = \theta\varepsilon, \tag{4.5}$$

where θ represents the elastic modulus or Young's modulus. Assume that $\{\Theta_i \colon i \geq 1\}$ are nonnegative r.v.'s. The tensile force function for this case is

$$f_\varepsilon(X_i) = (\Theta_i\varepsilon)\mathbf{1}_{(\varepsilon,\infty)}(\xi_i),$$

where $X_i = (\Theta_i, \xi_i)$ is a random vector with c.d.f. P. The sequence of i.i.d. random vectors $\{X_i \colon i \geq 1\}$ has components that are dependent, since ξ_i and Θ_i are both material properties.

Assume that ξ_i has a finite $2+\tau$, $\tau > 0$, moment and that Θ_i is bounded by $C < \infty$ so that the Lipschitz condition in (2.2) is satisfied. If ρ is continuous, then Theorem 2.1 guarantees the convergence in distribution of W_n to W. Clearly, for this specific example, the envelope for $\{f_\varepsilon \colon \varepsilon \text{ in } \mathbb{R}^+\}$ is in $\mathscr{L}^2(P)$ since Θ_i is bounded and ξ_i has a finite $2+\tau$, $\tau > 0$, moment. Consequently, $\{f_\varepsilon \colon \varepsilon \text{ in } \mathbb{R}^+\}$ is a special class of polynomial functions for which the metric entropy and VC results both hold; however, it is one of the simplest classes of such functions.

A straightforward generalization of (4.5) is

$$q(\varepsilon, \theta) = q_1(\theta) q_2(\varepsilon). \tag{4.6}$$

In order to apply the VC theory and the obvious generalization of Theorem 3.1, q_1 must be bounded and q_2 must be continuous and bounded. The product in (4.6) need not be a polynomial in ε and θ so that this case is distinctive from the polynomial behavior studied above. Only continuity is necessary for ξ_i. The class of functions $\mathscr{F}$ for this application is given by

$$\mathscr{F} = \{(s, t) \to q_1(s) q_2(\varepsilon) \mathbf{1}_{(\varepsilon,\infty)}(t) \colon \varepsilon \text{ in } \mathbb{R}^+\}.$$

The collection of graphs formed by $\mathscr{F}$ is the collection of *generalized wedges* truncated from below with respect to ε and with infinite extent in the directions of increasing t and s which, again, is a VC class of sets. By Lemma 2.5(iii), the assumption that q_2 is bounded and continuous can be relaxed to the asumption that q_2 is increasing, continuous, and in $\mathscr{L}^2(\rho)$ where ρ is the marginal c.d.f. for ξ_i.

In order to apply the metric entropy result of Theorem 2.1, q must be Lipschitz in the first variable. For (4.6) this implies that q_1 must be bounded and q_2 must be Lipschitz. Also, ξ_i must have a finite $2+\tau$, $\tau > 0$, moment and be continuous.

Elastic-plastic behavior

The tensile force function for this application is

$$f_\varepsilon(X_i) = \Theta_i \varepsilon \mathbf{1}_{(\varepsilon,\infty)}(\eta_i) + \Theta_i \eta_i \mathbf{1}_{[0,\varepsilon]}(\eta_i) \mathbf{1}_{(\varepsilon,\infty)}(\xi_i),$$

where $\{X_i = (\eta_i, \Theta_i, \xi_i) \colon i \geq 1\}$ is a collection of i.i.d. random vectors. This force function is called elastic-plastic because it is linear or elastic with a random modulus of Θ_i up to the yield strain η_i and then is perfectly plastic or constant until the random breaking strain ξ_i is reached. These components are quite likely to be dependent because all three represent material parameters. This is the simplest form of the load-strain function which has both the elastic and plastic behavior. Thus, it is frequently used in engineering applications, especially for approximations and bounds on the true material behavior. Clearly, it is a special case of (4.4).

Bilinear behavior

In engineering circles, the approximation most frequently used, if the elastic-plastic model is inappropriate , is

$$\begin{aligned} f_\varepsilon(X_i) = {} & \Theta_i \varepsilon \mathbf{1}_{(\varepsilon,\infty)}(\eta_i) + [\Phi_i \varepsilon + (\Theta_i - \Phi_i)\eta_i] \mathbf{1}_{[0,\varepsilon]}(\eta_i) \mathbf{1}_{(\varepsilon,\infty)}(\zeta_i) \\ & + [\Phi_i \zeta_i + (\Theta_i - \Phi_i)\eta_i] \mathbf{1}_{[0,\varepsilon]}(\eta_i) \mathbf{1}_{(\eta_i,\varepsilon)}(\zeta_i) \mathbf{1}_{(\varepsilon,\infty)}(\xi_i), \end{aligned}$$

where $\{X_i = (\eta_i, \Theta_i, \Phi_i, \zeta_i, \xi_i) \colon i \geq 1\}$ is a collection of i.i.d. random vectors. The elastic modulus is Θ_i, and the slope of the second linear portion is Φ_i. Both η_i and ζ_i are the strains at which transition occurs. As elsewhere , ξ_i is the breaking strain. Again, this is an obvious specialization of (4.4), and the convergence in distribution of W_n to W follows when the conditions of Lemmas 2.5 and 2.6 are met. Furthermore, this result can be extended to allow for multi-linear representations of the load-strain function.

There are several other generalizations which could be included. However, the major aplications have been mentioned above, and there is nothing mathematically new or interesting which can be contributed from other examples. As a final comment on applications, Phoenix (1979) has generalized the basic model for random slack to allow for mild dependency within subbundles of fibers and to allow for twisted cables. In each case, the restrictions on q and P are similar to those mentioned above, and they can be relaxed by a proof analogous to Theorem 2.1 or Theorem 3.1.

There are applications for which the bundle size n is very large, and the results herein are quite useful; however, in several physical problems n is too large for the c.d.f. of the bundle strength Q_n^* to be computed exactly but too small for the limiting c.d.f. to be tight. Smith (1982), by recasting the classical problem as a quantile process obtained by inverting the empirical distribution function for fiber strength, showed that the rate of convergence is essentially $O(n^{-1/3})$. Daniels (1989) used a heuristic argument based on the first passage time of Brownian motion and related processes across a parabolic boundary to reproduce Smith's (1982) result. Furthermore, he obtained similar results for the rates of convergence for two simplifications of the examples given in Phoenix and Taylor (1973). The remaining task, which will be the topic of another paper, is to study the rate of convergence in the context of empirical processes in order to allow for the more general examples presented herein.

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